Static Gauge and Energy Spectrum of Single-mode Strings in $AdS_5 \times S^5$

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Contents

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The AdS/CFT correspondence is one of the most fruitful research subject in the modern theoretical and mathematical physics.

It started with the work of Maldacena and states the equivalence between $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory in four-dimensional Minkowski space and superstring theory in the $AdS_5 \times S^5$ background.

Mapping the two sides to each other, weak coupling on one side is related to strong coupling on the other side. This makes the correspondence difficult to prove. But taken it for granted, one gains a very powerful tool for handling nonperturbative effects of string theory and gauge theory.

Another essential property of the correspondence is its holographic nature, i.e. a theory in a bulk is related to a theory on the boundary.

The correspondence has also been extended to other pairs of string and field theories including field theories with less supersymmetry and phenomenologically motivated string backgrounds.

Further applications concern the holographic description of strongly coupled systems typical for condensed matter physics.

Up to now there exists no equivalence proof and the correspondence is realized via a set of mapping rules for certain classes of quantities describing the partner theories.

One of such rules is the map of conformal scaling dimensions of composite operators of the gauge theory to the energy spectrum of certain string configurations.

The dual of string spectrum in $AdS_5 \times S^5$ are scaling dimensions of local operators of 4d $\mathcal{N} = 4$ super Yang-Mills theory. The first representative is the length-two Konishi operator

$$\mathrm{Tr}(\Phi_I(x)\Phi_I(x))$$

For the Konishi operator the anomalous scaling dimension has been computed for small coupling λ in perturbation theory up to 5-loops.

The assumption of integrability is powerful enough to evaluate the Konishi scaling dimension to even higher orders with the present record being set at eight or even nine loops.

Therefore, the computation of the string spectrum in $AdS_5 \times S^5$ became a classic problem in the AdS/CFT correspondence.

String dynamics in static gauge

Polyakov action

$$S = -\frac{T}{4\pi}\int \mathsf{d}\tau\,\mathsf{d}\sigma\ \sqrt{-h}\,h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,G_{\mu\nu}(x)$$

In the first order formalism this action is equivalent to

$$S = \int \mathsf{d}\tau\frac{\mathsf{d}\sigma}{2\pi}\left(P_\mu\,\dot{X}^\mu + \frac{\xi_1}{2T}\left(G^{\mu\nu}\,P_\mu P_\nu + T^2 G_{\mu\nu}\,X'^{\,\mu}X'^{\,\nu}\right) + \xi_2\ P_\mu X'^\mu\right)$$

Here, P_μ are the momentum variables conjugated to X^μ,
ξ_1, ξ_2 play the role of Lagrange multipliers.

Their variations provide the Virasoro constraints

$$G^{\mu\nu}\,P_\mu P_\nu + T^2 G_{\mu\nu}\,X'^{\,\mu}X'^{\,\nu} = 0\ , \qquad P_\mu X'^{\,\mu} = 0$$

A static spacetime metric tensor

$$G_{\mu\nu} = \begin{pmatrix} -\Lambda & 0 \\ 0 & G_{kl} \end{pmatrix}$$

with $\Lambda > 0$ and positive definite G_{kl} $(k, l = 1, 2, ..., D-1)$,
both X^0-independent.

From Neother theorem one gets conserved energy

$$E = -\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \; P_0$$

The static (temporal) gauge is defined by the gauge fixing conditions

$$X^0 = -\alpha P_0 \, \tau \ , \qquad P_0'(\tau, \sigma) = 0$$

with a positive constant α.

In the static gauge the first order action reduces to

$$S=\int \mathsf{d}\tau \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\,\left(P_k\,\dot{X}^k-\frac{\alpha P_0^2}{2}\right)$$

The square of energy density is obtained from the first Virasoro constraint

$$P_0^2=\Lambda\Big(G^{kl}P_kP_l+T^2\,G_{kl}\,X'^{\,k}X'^{\,l}\Big)$$

This Hamiltonian system has to be further reduced to the constraint surface

$$\mathcal{V}(\sigma)\equiv P_k(\sigma)X'^{\,k}(\sigma)=0 \qquad \mathcal{H}'(\sigma)=0\ ,$$

$$E^2=\frac{2H}{\alpha}=\int_0^{2\pi}\frac{\mathsf{d}\sigma}{2\pi}\,\Lambda\Big(G^{kl}P_kP_l+T^2\,G_{kl}\,X'^{\,k}X'^{\,l}\Big)$$

To analyze the constraints we calculate the Poisson brackets

$$\{\mathcal{V}(\sigma), \mathcal{V}(\tilde{\sigma})\} = \left[\mathcal{V}'(\sigma)\, \delta(\sigma - \tilde{\sigma}) + 2\mathcal{V}(\sigma)\, \delta'(\sigma - \tilde{\sigma})\right] \ ,$$

$$\{\mathcal{V}(\sigma), \mathcal{H}(\tilde{\sigma})\} = \left[\mathcal{H}'(\sigma)\, \delta(\sigma - \tilde{\sigma}) + 2\mathcal{H}(\sigma)\, \delta'(\sigma - \tilde{\sigma})\right] \ ,$$

$$\{\mathcal{H}(\sigma), \mathcal{H}(\tilde{\sigma})\} = \alpha^2[(\Lambda^2(\sigma)\mathcal{V}(\sigma))'\, \delta(\sigma - \tilde{\sigma}) + 2\Lambda^2(\sigma)\mathcal{V}(\sigma)\, \delta'(\sigma - \tilde{\sigma})]$$

From these brackets follow that the constraints are of the second class and they are preserved in dynamics

$$\{H, \mathcal{V}(\sigma)\} = \mathcal{H}'(\sigma) \ , \qquad \{H, \mathcal{H}'(\sigma)\} = \alpha^2(\Lambda^2(\sigma)\mathcal{V}(\sigma))''$$

Construction of independent canonical variables on the constraint surface is not an easy task even for Minkowski space with $G_{kl} = \delta_{kl}$ and $\Lambda = 1$.

In this case the system describes $(D-1)$ massless free fields in two dimensions and the constraints (1) correspond to $L_n = 0 = \bar{L}_n$ for $n \neq 0$, where L_n and $\bar{L}_n$ are the standard Virasoro generators of 2d free CFT.

It is therefore easier to first quantize the free-field theory and then take into account the constraints on the quantum level by the equations for the physical states $L_n|\psi_{ph}\rangle = 0 = \bar{L}_n|\psi_{ph}\rangle$ $(n > 0)$.

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$AdS_5 \times S^5$ string in static gauge

AdS_5 is realized as a 5-dimensional hyperbola $\mathcal{Z}_A \mathcal{Z}^A = -R^2$ in $\mathbb{R}^{2,4}$ $A = 0, 0', 1, ..., 4$.

S^5 as 5-dimensional sphere $\mathcal{Y}_I \mathcal{Y}^I = R^2$ in $\mathbb{R}^6$, $I = 1, ..., 6$.

We parameterize the embedding coordinates of $\mathbb{R}^{2,4}$ by

$$\mathcal{Z}^{0'} = R\sqrt{1+\vec{X}^2}\,\sin(X^0)\ , \quad \mathcal{Z}^{0} = R\sqrt{1+\vec{X}^2}\,\cos(X^0)\ , \quad \mathcal{Z}^{a} = R\,X^a$$

where X^0 is the AdS_5 time coordinate and $\vec{X}^2 \equiv X^b X^b$.

For the spherical part one can use the coordinates of the stereographic projection

$$\mathcal{Y}^i = \frac{R\,Y^i}{1+\vec{Y}^2/4}\ , \qquad \mathcal{Y}^6 = R\,\frac{1-\vec{Y}^2/4}{1+\vec{Y}^2/4}\ ,$$

with $\vec{Y}^2 \equiv Y^j Y^j$ $(i, j = 1, \ldots, 6)$.

The induced metric on $AdS_5 \times S^5$ takes the following block structure

$$g_{\mu\nu} = \begin{pmatrix} -\Lambda & 0 & 0 \\ 0 & G_{ab} & 0 \\ 0 & 0 & G_{ij} \end{pmatrix} \ , \qquad \text{where}$$

$$\Lambda = R^2(1+\vec{X}^2) \ , \quad G_{ab} = R^2 \left(\delta_{ab} - \frac{X^a\,X^b}{1+\vec{X}^2}\right) \ , \quad G_{ij} = \frac{R^2\,\delta_{ij}}{(1+\vec{Y}^2/4)^2}$$

and the corresponding inverse matrices read

$$G^{ab} = \frac{1}{R^2}\left(\delta_{ab} + X^a\,X^b\right) \ , \qquad G^{ij} = \frac{1}{R^2}\left(1+\vec{Y}^2/4\right)^2 \delta_{ij} \ .$$

Applying the static gauge with $\alpha = 1/(R^2 T)$, we obtain

$$E^2 = \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\ (1{+}\vec{X}^2) \left[\vec{P}^{\,2} + (\vec{P}\cdot\vec{X})^2 + \lambda \left(\vec{X}'^{\,2} - \frac{(\vec{X}\cdot\vec{X}')^2}{1+\vec{X}^2} \right) + \mathcal{M}_S^2 \right] \ ,$$

where $\lambda \equiv R^4\, T^2$ and $\mathcal{M}_S^2$ denotes the spherical part

$$\mathcal{M}_S^2 = (1+\vec{Y}^2/4)^2\, \vec{P}_Y^2 + \lambda\, \frac{\vec{Y}'^{\,2}}{(1+\vec{Y}^2/4)^2} \ .$$

Analyzing this system one has to recall that the integrand is σ-independent and, in addition, the level matching density vanishes

$$\mathcal{V} = \vec{P}\cdot\vec{X}' + \vec{P}_Y \cdot \vec{Y}' = 0$$

The dynamical integrals related to the rotations in $\mathbb{R}^{2,4}$ and $\mathbb{R}^6$,

$$J_{AB} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\ V^{\mu}_{AB} P_{\mu}\ , \qquad L_{IJ} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\ V^{\mu}_{IJ} P_{\mu}\ , \tag{1}$$

generate the isometry group $\mathrm{SO}(2,4)\times\mathrm{SO}(6)$.

The index μ in (1) incorporates the three blocks $\mu = (0,\, a,\, 4+i)$,

$$P_0 = -E$$

V^{μ}_{AB}, V^{μ}_{IJ} are the components of the Killing vector fields in $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$V^0_{AB} = G^{00}(\mathcal{Z}_B\partial_0\mathcal{Z}_A - \mathcal{Z}_A\partial_0\mathcal{Z}_B)\ , \qquad V^a_{AB} = G^{ab}(\mathcal{Z}_B\partial_b\mathcal{Z}_A - \mathcal{Z}_A\partial_b\mathcal{Z}_B)\ ,$$

$$V^{i+N}_{IJ} = G^{ij}(\mathcal{Y}_J\partial_j\mathcal{Y}_I - \mathcal{Y}_I\partial_j\mathcal{Y}_J)\ .$$

The one-mode ansatz

$$\vec{X}(\tau,\sigma) = \vec{X}_0(\tau) + \vec{X}_+(\tau)\, e^{im\sigma} + \vec{X}_-(\tau)\, e^{-im\sigma} \qquad \vec{Y}(\tau,\sigma) = \vec{Y}(\tau)\ ,$$
$$\vec{P}(\tau,\sigma) = \vec{P}_0(\tau) + \vec{P}_+(\tau)\, e^{im\sigma} + \vec{P}_-(\tau)\, e^{-im\sigma} \qquad \vec{P}_Y(\tau,\sigma) = \vec{P}_Y(\tau)\ .$$

$$\mathcal{M}_S^2 = (1+\vec{Y}^2/4)\vec{P}_Y^2 = \frac{1}{2}\, L_{ij}L_{ij} + L_{i\,M+1}L_{i\,M+1}\ .$$

To preserve the one-mode ansatz in dynamics, the scalar combinations

$$\vec{X}^2\ ,\quad \vec{P}^2\ ,\quad \vec{P}\cdot\vec{X}, \ldots$$

have to be σ-independent.

We find the following conditions on the exited modes

$$\vec{X}_{\pm}^2 = \vec{P}_{\pm}^2 = 0 \ , \quad \vec{X}_{\pm} \cdot \vec{X}_0 = \vec{P}_{\pm} \cdot \vec{P}_0 = 0 \ , \quad \vec{P}_{\pm} \cdot \vec{X}_{\pm} = 0 \ ,$$

$$\vec{X}_{\pm} \cdot \vec{P}_0 = \vec{P}_{\pm} \cdot \vec{X}_0 = 0 \ , \quad \vec{P}_{+} \cdot \vec{X}_{-} - \vec{P}_{-} \cdot \vec{X}_{+} = 0 \ .$$

Introducing real and imaginary parts of non-zero modes,

$$\vec{X}_{\pm} = \frac{1}{\sqrt{2}}(\vec{X}_{\rm re} \pm i\vec{X}_{\rm im}) \ , \qquad \vec{P}_{\pm} = \frac{1}{\sqrt{2}}(\vec{P}_{\rm re} \pm i\vec{P}_{\rm im})$$

Non-zero mode vectors can be parameterized in the form

$$\vec{X}_{\rm re} = \frac{Q}{\sqrt{2}}\,\vec{\mathbf{e}}_{\rm re} \ , \quad \vec{P}_{\rm re} = \frac{P}{\sqrt{2}}\,\vec{\mathbf{e}}_{\rm re} \ ,$$
$$\vec{X}_{\rm im} = \frac{Q}{\sqrt{2}}\,\vec{\mathbf{e}}_{\rm im} \ , \quad \vec{P}_{\rm im} = \frac{P}{\sqrt{2}}\,\vec{\mathbf{e}}_{\rm im}$$

String surface

$$\vec{X}(\tau,\sigma) = \vec{X}_0(\tau) + Q(\tau)\left[\vec{\mathbf{e}}_1 \cos(m\sigma+\varphi) + \vec{\mathbf{e}}_2 \sin(m\sigma+\varphi)\right]$$

Physical phase space coordinates

$$(\vec{P}_0,\, \vec{X}_0,\, P,\, Q)$$

The pair $(P,\, Q)$ is given on the half-plane $Q \geq 0$.

Canonical symplectic form

$$\Omega = \int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi}\, \mathsf{d}\vec{P}(\sigma) \wedge \mathsf{d}\vec{X}(\sigma) = \mathsf{d}\vec{P}_0 \wedge \mathsf{d}\vec{X}_0 + \mathsf{d}P \wedge \mathsf{d}Q$$

Energy square and the isometry group dynamical integrals

$$E^2 = \left(1 + \vec{X}_0^2 + Q^2\right)\left(\vec{P}_0^{\,2} + P^2 + (\vec{P}_0 \cdot \vec{X}_0 + PQ)^2 + \mathcal{M}_S^2 + \lambda m^2 Q^2\right) \ ,$$

$$J_{0'\,0} = E \ , \qquad J_{a\,b} = P_{0\,,a}\, X_0^b - P_{0\,,b}\, X_0^a \ ,$$

$$\vec{J}_{0'} = \frac{E\,\vec{X}_0}{\sqrt{1 + \vec{X}_0^2 + Q^2}} \ , \qquad \vec{J}_{0'} = -\sqrt{1 + \vec{X}_0^2 + Q^2}\ \vec{P}_0$$

The SO(2,4) Casimir

$$C = (P + \tilde{D}\, Q)^2 + (1 + Q^2)(\mathcal{M}_S^2 + \lambda m^2\, Q^2) \ ,$$

where $\tilde{D} = P\,Q + \vec{P}_0 \cdot \vec{X}_0$ is the generator of dilatations.

Canonical transformation of the phase space variables

$$\vec{P}_0 = \frac{\vec{p}_{_0}}{\sqrt{1+q^2}} \ , \quad \vec{X}_0 = \vec{x}_{_0}\sqrt{1+q^2} \ , \quad P = p - \frac{q(\vec{p}_{_0}\cdot\vec{x}_{_0})}{1+q^2} \ , \quad Q = q \ .$$

Dynamical integrals in the new variables take the form of a massive particle

$$E^2 = \left(1+\vec{x}_{_0}^{\,2}\right)\left(\vec{p}_{_0}^{\,2} + (\vec{p}_{_0}\cdot\vec{x}_{_0})^2 + \mathcal{M}^2\right) \ ,$$

$$J_{0'\,0} = E \ , \qquad\qquad J_{a\,b} = p_{_{0\,,a}}\, x_{_0}^b - p_{_{0\,,b}}\, x_{_0}^a \ ,$$

$$\vec{J}_{0'} = \frac{E\,\vec{x}_{_0}}{\sqrt{1+\vec{x}_{_0}^{\,2}}} \ , \qquad \vec{J}_0 = -\sqrt{1+\vec{x}_{_0}^{\,2}}\ \vec{p}_{_0}.$$

The mass-square term depends only on the coordinates $(p,\, q)$

$$\mathcal{M}^2 = (1+q^2)\left[p^2 + p^2q^2 + \mathcal{M}_S^2 + \lambda m^2 q^2\right] \ .$$

Calculation of the energy spectrum

The minimal energy of the AdS_5 particle with Casimir number $\mathcal{M}^2$ reads

$$E_0 = 2 + \sqrt{\mathcal{M}^2 + 4}\ .$$

Thus, to find the energy levels of the pulsating string in the rest frame, one has to calculate the spectrum of the operator $\mathcal{M}^2$.

Note that the eigenvalues of the SO(6) Casimir number are $J(J+4)$ with non-negative integer J and we replace $\mathcal{M}_S^2$ by $J(J+4)$.

Quantizing the system on the half-plane $(p,\, q)$, one has to choose either Dirichlet or Neumann boundary conditions for wave functions $\psi(q)$ at $q = 0$. This ambiguity is related to the non-selfadjointness of the momentum operator p on the half-line $q \geq 0$.

Supersymmetric correction

$$\mathcal{M}^2 \mapsto \mathcal{M}^2 - 8\mathcal{M}$$

N_q dimensional model

$$\mathcal{M}^2 = (1 + \vec{q}^2) \left[\vec{p}^2 + (\vec{p} \cdot \vec{q})^2 + J(J+4) + \lambda m^2 \vec{q}^2 \right] \ .$$

Here $\vec{p}$ and $\vec{q}$ are N_q dimensional vectors.
We gauge the system by the O(N_q) group.

The rescaling of the phase space variables

$$q \mapsto \frac{q}{\lambda^{1/4} \, m^{1/2}} \ , \qquad p \mapsto \lambda^{1/4} \, m^{1/2} \, p \ ,$$

yields the mass-square $\mathcal{M}^2$ in the form

$$\mathcal{M}^2 = \sqrt{\lambda} \, m \left(\vec{p}^2 + \vec{q}^2 \right) + \vec{p}^2 \, \vec{q}^2 + (\vec{p} \cdot \vec{q})^2 + (\vec{q}^2)^2 + J(J+4) \ .$$

To mimic the supersymmetry one applies the normal order prescription and then takes $N_q = 0$ in the final expressions.
We find the mean values

$$\langle n|\mathcal{M}^2|n\rangle = 4\lambda^{1/2}\, mn + 10n^2 + (N_q - 6)n + J(J+4)$$

and for $\mathcal{N} = 0$ they yield

$$\langle n|\mathcal{M}^2|n\rangle = 4\lambda^{1/2} mn + 10n^2 - 6n + J(J+4)\ .$$

Equation (3) then leads to the following energy spectrum

$$E_{m,n}(J) = 2\lambda^{1/4}\sqrt{mn} - 2 + \frac{10n^2 - 6n + 4 + J(J+4)}{4\lambda^{1/4}\sqrt{mn}} + \dots\ ,$$

which for $m = 1$, $n = 1$ and $J = 0$ reduces to

$$E_{1,1} = 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \dots$$

Outlook

The expansion of the Hamiltonian in powers of $1/\sqrt{\lambda}$

$$H = H^{(0)} + \frac{1}{\sqrt{\lambda}} H^{(1)} + \frac{1}{\lambda} H^{(2)} + \dots \ ,$$

is obtained if one uses the rescaled phase space coordinates

$$x^k = \lambda^{1/4} X^k \ , \qquad p_k = \lambda^{-1/4} P_k \ ,$$

where $X^k := (X^a, Y^i)$, $P_k := (P_a, P_{Y^i})$ $(k = 1, 2, \dots, 9)$. The leading term here coincides with the static gauge string Hamiltonian in 10-dimensional Minkowski space

$$H^{(0)} = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \left(p^2(\sigma) + x'^{\,2}(\sigma) \right) \ .$$

The rotation generators J_{ab} and J_{ij} are invariant under the rescaling and, therefore, they are λ-independent.

The generators $J_{a0'}$ are expanded in powers of $1/\sqrt{\lambda}$ like the Hamiltonian. The leading terms of their expansion correspond to four boosts in x^a $(a = 1, \ldots, 4)$ directions of the 10-dimensional Minkowski space. Other symmetry generators are singular at $\lambda \to \infty$, however, after their rescaling by the factor $\lambda^{-1/4}$, they also become analytic in $1/\sqrt{\lambda}$. It is easy to check that the corresponding leading terms define the translation generators of 10-dimensional Minkowski space.

With a supersymmetric extension, the leading order part is quantized without anomalies. This part of the symmetry generators and the commutation relations of the isometry group can be used as a basis for perturbative quantum calculations.