Analysis of the τ-lepton decay data within a dispersive approach to perturbative QCD

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● τ system provides a unique possibility to determine fundamental as well as phenomenological parameters of QCD (see seminal work and the literature therein).
E. Braaten, S. Narison, A. Pich, Nucl. Phys. B **373** (1992) 581.

The central theoretical quantity that describes τ-lepton hadronic decays is spectral density of hadronic states related with the two point correlator of hadronic currents, in the chiral limit

$$\Pi_{\mu\nu}(q^2) = \imath \int d^4x < 0|T(V_\mu(x)V_\nu(0))|0 >= (-g_{\mu\nu}q^2+q_\mu q_\nu)\Pi(q^2), \quad (1)$$

The correlator is approximated in the QCD perturbation theory augmented with Operator Product Expansion (OPE)

M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **147** (1979) 385.

The OPE is parametrized in terms of the vacuum expectation values of operators, the condensates. A reasonable description of the τ-lepton decays implies a careful evaluation of numerical values for these phenomenological parameters.

A powerful method to evaluate QCD predictions for the τ decays is the Finite Energy Sum Rules (FESR)
N.V. Krasnikov, A.A. Pivovarov, A.N. Tavkhelidze, Z. Phys. C **19** (1983) 301.

combined with the renormalization group (RG) technique. However, the RG invariance and the FESR cannot be combined unambiguously. The most popular methods are fixed order perturbation theory (FOPT) and contour improved perturbation theory

(CIPT)
A.A. Pivovarov, Z. Phys. C **53** (1992) 461
F. Le Diberger, A Pich Phys. Lett. B **286** (1992) 147.

In the literature, arguments have been given that **CIPT is conceptually inconsistent**: the predictions obtained within CIPT may be distorted due to the non-physical Landau singularities which present in the running coupling.

The Kallen-Lehmann Analyticity of the correlators is a strong consequence of the general principles of the QFT.

To overcome this obstacle analytic or dispersive approaches to perturbative QCD are being developed

The most prominent analytic approach: the Analytic Perturbation Theory (APT)
D.V. Shirkov, I.L. Solovtsov, Phys. Rev. Lett. **79** (1997) 1209.

K.A. Milton, I.L. Solovtsov, O.P. Solovtsova, Phys. Lett. **B415** (1997) 104.

However, it was shown in the work
B.V. Geshkenbein, B.L. Ioffe and K.N. Zyablyuk, Phys.Rev. D64: 093009,2001
That APT with massless quarks is in strong contradiction with experiment.

More serious difficulty is that APT violates the OPE. It predicts new power suppressed contributions of ultraviolet origin in the correlators, that are not included in OPE. At present the experiment does not confirm the presence of such terms.

A specific dispersive framework to analyze the hadronic τ data from the vector non-strange channel, referred to as Dispersive Contour Improved Perturbation Theory(DCIPT) developed in:
B.A. Magradze, Few-Body Syst. **48**/2-4 (2010) 143 [Erratum-ibid.**53**/3, (2012) 365. in
B. Magradze, Proceedings of A. Razmadze Mathematical Institute **160**, (2012) 111.

Analyticity is imposed only on the physical quantities (correlators) and modification of the running coupling is not considered.

we have consistently extracted from the data reasonable values for the strong coupling constant $\alpha_s(m_\tau)$ and the continuum threshold s_1, the energy squared parameter that defines in the Minkowski space the lower bound of the quark-hadron duality.

In the present paper, we employ the same framework, to extract the numerical values for the QCD vacuum condensates from the V-channel τ data.

We consider the correlator of hadronic vector currents, $V_\mu = \bar{u}\gamma_\mu d$, in the chiral limit

$$\Pi_{\mu\nu}(q^2) = \imath \int d^4x < 0|T(V_\mu(x)V_\nu(0))|0 >= (-g_{\mu\nu}q^2+q_\mu q_\nu)\Pi(q^2), \tag{2}$$

the spectral density of hadronic states is related to the absorptive part of the correlator

$$v_1(s) = 2\pi \mathrm{Im}\Pi(s+\imath 0), \tag{3}$$

with $s = q^2$. The correlator has the asymptotic expansion in the

Euclidean domain, the OPE,

$$\Pi(-Q^2) = \sum_{\mathbf{d}=0,2,\ldots} \frac{\mathcal{O}_{\mathbf{d}}(Q^2)}{(Q^2)^{\mathbf{d}/2}}, \tag{4}$$

where $Q^2 = -q^2$, $\mathcal{O}_{\mathbf{d}}(Q^2) = \mathcal{C}_{\mathbf{d}}(Q^2) < \mathcal{O}_{\mathbf{d}} >$, $< \mathcal{O}_{\mathbf{d}} >$ is the QCD non-perturbative condensate of dimension $\mathbf{d}$ and $\mathcal{C}_{\mathbf{d}}(Q^2)$ is the associated Wilson coefficient function. The term of the lowest dimension $\mathbf{d = 0}$, the pure perturbative component of the correlator, is usually separated in (4)

$$\Pi(-Q^2) = \Pi_{\mathsf{PT}}(-Q^2) + \Pi_{\mathsf{OPE}}(-Q^2). \tag{5}$$

It is enough to consider the Adler function, the physical quantity, related to the correlator

$$D(Q^2) = -(4\pi^2)Q^2 \frac{d}{Q^2}\Pi(-Q^2) = D_{\mathsf{PT}}(Q^2) + D_{\mathsf{OPT}}(Q^2), \tag{6}$$

the perturbative component of the Adler function $D_{\mathrm{PT}}(Q^2)$ has been calculated in perturbation theory up to terms of order α_s^4, (i.e. at N^{3}LO). Using the RG invariance, the perturbation theory series for $D_{\mathrm{PT}}(Q^2)$ can be rewritten as the series in powers of the running coupling $\alpha_s(Q^2)$

$$D_{\mathrm{PT}}(Q^2) = \sum_{k=0}^{\infty} d_k \left(\frac{\alpha_s(Q^2)}{\pi}\right)^k, \tag{7}$$

in the $\overline{\mathrm{MS}}$ scheme for $n_f = 3$ quark flavours the known coefficients of the series take the values

$$d_0 = d_1 = 1,\ d_2 = 1.6398,\ d_3 = 6.3710,\ d_4 = 49.0757.$$

The exact Adler function satisfies the dispersion relation (DR)

$$D(Q^2) = Q^2 \int_0^\infty \frac{2v_1(s)ds}{(s+Q^2)^2}, \tag{8}$$

the inversion of (8) reads

$$v_1(s) = \frac{1}{4\pi\imath}\oint_{-s-\imath\epsilon}^{-s+\imath\epsilon} \frac{D(z)}{z} d\,z, \tag{9}$$

where the path of integration, connecting the points $-s\mp\imath\epsilon$ in the complex z-plane, avoids the cut running along the real negative z axis. It follows from the asymptotic expansion (7) that the exact function $\mathrm{d}(Q^2) = D(Q^2) - 1$ satisfies an unsubtracted DR

$$\mathrm{d}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{\rho_{\mathrm{eff}}(\sigma)}{\sigma + Q^2} d\,\sigma, \tag{10}$$

with the effective spectral density determined as

$$\rho_{\mathrm{eff}}(\sigma) = \mathrm{Im}\{\mathrm{d}(-\sigma - \imath\epsilon)\}. \tag{11}$$

From the above given formulas follows a simple relation

$$v_1(s) = \frac{1}{2}(1 + r(s)) \quad \text{where} \quad r(s) = \frac{1}{\pi}\int_s^\infty \frac{\rho_{\mathrm{eff}}(\sigma)}{\sigma} d\,\sigma. \tag{12}$$

The quark-hadron duality can be described via the following *ansatz* for the hadronic spectral function

$$v_1(s) \simeq v_1(s)|_{\text{s.exp}} = \theta(s_1 - s) v_1(s)|_{\text{exp}} + \theta(s - s_1) v_1(s)|_{\text{pQCD}}, \quad (13)$$

where the subscript "s.exp" stands for "semi-experimental", s_1 denotes the continuum threshold, the energy squared above which we trust perturbative QCD, we assume that

$$0 < s_1 < m_\tau^2,$$

$v_1(s)|_{\text{exp}}$ is the spectral function measured on the experiment and $v_1(s)|_{\text{pQCD}}$ is determined by the theoretical model, i.e. QCD. Using (13), we construct the "semi-experimental" Adler function

$$D(Q^2) \simeq D(Q^2)|_{\text{semi.ex}} = D(Q^2, s_1)|_{\text{exp}} + D(Q^2, s_1)|_{\text{pQCD}}, \quad (14)$$

where we denote

$$D(Q^2, s_1)|_{\text{exp}} = Q^2 \int_0^{s_1} \frac{2 v_1(s)|_{\text{exp}} d\, s}{(s + Q^2)^2}, \quad (15)$$

$$D(Q^2, s_1)|_{\mathrm{pQCD}} = Q^2 \int_{s_1}^{\infty} \frac{2v_1(s)|_{\mathrm{pQCD}} d\, s}{(s+Q^2)^2} \qquad (16)$$

To evaluate the QCD component of the spectral function in formula (13), we insert the RG improved approximation (7) for the Adler function into inversion formula (9). Indeed, for sufficiently large values of $|s| \geq s_1$, it is reasonable to employ the asymptotic formula (4) on the integration contour of the integral (9). This resembles the CIPT approach to the τ-decays. The difference is that within our approach the approximation to the Adler function preserves the analyticity properties of the exact function. Let us ignore the perturbative logarithmic corrections to the Wilson coefficients. In this approximation, the non-perturbative component $D_{\mathrm{OPE}}(Q^2)$ of the Adler function will not contribute to the spectral function for $s > s_1$. In fact, its contribution has the form

$$v_1(s)|_{\mathrm{OPE}} = 2\pi^2 \sum_{k=1,2...} C_{2k} < \mathcal{O}_{2k} > \frac{k(-1)^{k-1}}{(k+1)(k-1)!}\delta^{(k-1)}(s). \tag{17}$$

Thus we deduce

$$v_1(s)|_{\mathrm{pQCD}} = v_1(s)|_{\mathrm{APT}}, \qquad s > s_1 \tag{18}$$

where $v_1(s)|_{\mathrm{APT}}$ is the spectral function calculated in the sense of the Shirkov-Solovtsov Analytic Perturbation Theory. In this special case, the effective spectral density is determined as

$$\rho_{\mathrm{eff}}(\sigma)|_{\mathrm{APT}} = \mathrm{Im}\left\{\sum_{n=1} d_n \left(\frac{\alpha_s(-\sigma - \imath 0))}{\pi}\right)^n\right\}. \tag{19}$$

In earlier works, we have derived the violated DR

$$\mathsf{d}(Q^2)|_{\mathsf{PT}} = \frac{1}{\pi}\int_0^\infty \frac{\rho_{\mathsf{eff}}(\sigma)|_{\mathsf{APT}}}{\sigma + Q^2} d\,\sigma + \mathsf{d}_{\mathsf{L}}(Q^2), \tag{20}$$

where

$$\mathsf{d}_{\mathsf{L}}(Q^2) = -\frac{1}{2\pi\imath}\oint \frac{\mathsf{d}(\zeta)|_{\mathsf{PT}}}{\zeta - Q^2} d\,\zeta, \tag{21}$$

the contour integral in the complex plane is taken round the circle $|\zeta - s_{\mathsf{L}}| = s_{\mathsf{L}}$ where s_{L} denotes the Landau singularity of the running coupling located on the positive Q^2-axis. An important issue is how to define the power suppressed part of the semi-experimental Adler function, our suggestion is

$$D(Q^2)|_{\mathsf{pow}} = D(Q^2)|_{\mathsf{semi.ex}} - D(Q^2)|_{\mathsf{PT}}, \tag{22}$$

the right hand side of (22) can be rewritten in the form

$$D(Q^2)|_{\mathrm{pow}} = 2\int_0^{s_1} K(Q^2,s)(v_1(s)|_{\mathrm{exp}} - v_1(s)|_{\mathrm{APT}})d\,s - \mathrm{d_L}(Q^2), \tag{23}$$

where $K(Q^2,s) = Q^2/(Q^2+s)$. Formula (23) enables us to write the asymptotic expansion at large positive Q^2

$$D(Q^2)|_{\mathrm{pow}} \simeq \sum_{k=1,2} \eta_{2k}\frac{\Lambda^{2k}}{Q^{2k}}. \tag{24}$$

where $\Lambda \equiv \Lambda_{\mathrm{QCD}}$ is the QCD scale parameter. On the other hand, for large positive Q^2 the asymptotic expansion of the "semi-experimental" Adler function should reproduce the OPE

$$D(Q^2)|_{\mathrm{s.exp}} \simeq D_{\mathrm{PT}}(Q^2) + D_{\mathrm{OPE}}(Q^2), \tag{25}$$

where

$$D_{\mathsf{OPE}}(Q^2) \simeq 2\pi^2 \sum_{\mathbf{d}=2,4...} \frac{\mathbf{d}\mathcal{O}_{\mathbf{d}}}{Q^{\mathbf{d}}}. \tag{26}$$

In the accepted approximation, the effective condensate combinations $\mathcal{O}_{\mathbf{d}}$ are independent of Q^2. At large space-like Q^2,

$$D_{\mathsf{OPE}}(Q^2) = D_{\mathsf{pow}}(Q^2)$$

This condition leads to the equation

$$\mathsf{d}_{\mathsf{L}}(z) + 2z\int_0^{s1} \frac{v_1(s)|_{\mathsf{APT}}}{(s+z)^2} d\,s + D_{\mathsf{OPE}}(z) = 2z\int_0^{s1} \frac{v_1(s)|_{\mathsf{exp}}}{(s+z)^2} d\,s, \tag{27}$$

where $z = Q^2 >> \Lambda^2$. Expanding Eq. (27) in powers of $1/z$ and equating the expansion coefficients we obtain the sum rules

$$\frac{1}{2\pi\imath}\oint_{|\zeta-s_{\mathsf{L}}|=s_{\mathsf{L}}} \mathsf{d}(\zeta)|_{\mathsf{PT}}\zeta^{k-1}d\,\zeta + 2(-1)^{k-1}k\int_0^{s1} s^{k-1}v_1(s)|_{\mathsf{APT}}d\,s$$
$$+4\pi^2 k\mathcal{O}_{2k} = 2(-1)^{k-1}k\int_0^{s1} s^{k-1}v_1(s)|_{\exp}d\,s, \qquad k=1,2\,. \tag{28}$$

these sum rules relate the parameter Λ, s_1 and the effective condensate combinations $\mathcal{O}_{2k}$ with the experimental data. The contour integral on the LHS of Eq. (28) may be written as

$$\frac{1}{2\pi\imath}\oint_{|\zeta-s_{\mathsf{L}}|=s_{\mathsf{L}}} \mathsf{d}(\zeta)|_{\mathsf{PT}}\zeta^{k-1}d\,\zeta = \Lambda^{2k}C_{\mathsf{L,k}}, \tag{29}$$

where $C_{\mathsf{L,k}}$ is a pure number which is calculated by solving the RG equation on the circle $|\zeta - s_{\mathsf{L}}| = s_{\mathsf{L}}$. The second integral on the LHS of Eq. (28) may be rewritten in terms of the effective spectral function by using the relation (12) and integrating by parts. Then the sum rules (28) take the following form

$$\Lambda^{2k} C_{\mathsf{L,k}} + (-1)^{k-1}\left(s_1^k(1+r(s_1)|_{\mathsf{APT}}) + \frac{1}{\pi}\int_0^{s_1} s^{k-1}\rho_{\mathsf{eff}}(s)|_{\mathsf{APT}}d\,s\right)$$
$$+4\pi^2 k\mathcal{O}_{2k} = 2(-1)^{k-1}k\int_0^{s_1} s^{k-1}v_1(s)|_{\mathsf{exp}}d\,s, \qquad k=1,2\,.... \tag{30}$$

In the chiral limit the dimension $\mathbf{d}=2$ operator vanishes. Thus Eq. (30) for $k=1$ involves only parameters s_1 and Λ. One more relation between the two parameters follows from the duality condition

$$\int_{s_1}^{s_2} w_\tau(s)v_1(s)|_{\mathsf{APT}}d\,s = \int_{s_1}^{s_2} w_\tau(s)v_1(s)|_{\mathsf{exp}}d\,s, \tag{31}$$

where $w_\tau(s) = \frac{1}{s_2}\left(1-\frac{s}{s_2}\right)^2\left(1+2\frac{s}{s_2}\right)$ with $s_2 = m_\tau^2$, m_τ being the τ lepton mass. Equation (31) and Eq.(30) for $k=1$ can be used to determine numerical values of the parameters s_1 and Λ from the data. In previous papers we have solved numerically this system of equations for these parameters using the 2005 ALEPH data as well as the 1998 OPAL data. Since then the ALEPH data has been corrected in 2008 (and in 2010). In the present paper we employ the corrected ALEPH data as well as updated values for the relevant to the τ decays parameters

$$m_\tau = 1776.9 \pm 0.2\,\text{MeV} \quad V_{\text{ud}} = 0.97418 \pm 0.00019 \quad S_{\text{EW}} = 1.0198 \pm 0.00$$

With the corrected data our previous estimates for the parameters somewhat changed. The new numerical values for the parameters are given in the Table.

Observable	Approximation to the Adler function	
	N^2LO	N^3LO
s_1 GeV^2	1.724 ± 0.053	1.721 ± 0.054
Λ MeV	343 ± 30	322 ± 27
$\alpha_s(m_\tau^2)$	0.315 ± 0.015	0.308 ± 0.014

The effective condensate combinations $\mathcal{O}_{2k}$, $k = 2, 3 \ldots$ are completely determined from the experimental data, once the values of the parameters s_1 and Λ are known. First we calculated the coefficients $C_{\mathrm{L,k}}$ which contribute in the sum rules (30). To do this we have solved the RG equation numerically at three- and four loop orders in the complex plane. In the calculations we have used the N^2LO and N^3LO approximations to the Adler function

$\mathbf{d} = 2k$	$C_{\mathsf{L,k}}\|_{\mathsf{N^2LO}}$	$C_{\mathsf{L,k}}\|_{\mathsf{N^3LO}}$
2	0.52726	0.65137
4	1.1055	2.0295
6	2.4314	6.5532
8	5.4125	21.048
10	12.104	67.219
12	27.124	213.84

$\mathbf{d}=2k$	$\mathcal{O}_\mathbf{d}\vert_{\mathsf{N^2LO}}$	$\mathcal{O}_\mathbf{d}\vert_{\mathsf{N^3LO}}$
4	2.91 ± 0.79	2.72 ± 0.82
6	-11.91 ± 1.38	-11.80 ± 1.37
8	21.68 ± 2.47	21.47 ± 2.49
10	-34.00 ± 4.29	-33.69 ± 4.30
12	51.1 ± 7.4	50.6 ± 7.4

In Table we present the numerical results for the condensate combinations $\mathcal{O}_\mathbf{d}$, for $\mathbf{d} = 4 - 12$, extracted from the ALEPH data. The condensate combinations are given in units $10^{-3}\mathrm{GeV}^\mathbf{d}$. The calculations are performed using the N^2L0 and N^3L0 approximations to the Adler function. The experimental errors are indicated only.

To determine the experimental uncertainties on the extracted numerical values of the condensate combinations we have used the covariance matrixes for invariant mass squared distributions provided by ALEPH. Comparing the central values at N^2LO and N^3LO orders in the Table, we see that the theoretical errors due to truncation of the perturbation theory series is very small. However, the total theoretical errors should be significantly large. The work in this direction is in progress.

We remark that our values, for the operators $\mathcal{O}_4$, $\mathcal{O}_6$ and $\mathcal{O}_8$, actually agree well with those of

C.A. Dominguez, K. Schilcher, JHEP **0701** 093, (2007).

These authors analyzed the V-channel ALEPH data within FOPT and CIPT.

Operator	numerical value
$\mathcal{O}_4\|_{\rm FOPT}$	$(2.53 \pm 1.27) * 10^{-3}\rm GeV^4$
$\mathcal{O}_4\|_{\rm CIPT}$	$(3.80 \pm 1.01) * 10^{-3}\rm GeV^4$
$\mathcal{O}_4\|_{\rm DIPT}$	$(2.72 \pm 0.82) * 10^{-3}\rm GeV^4$
$\mathcal{O}_6\|_{\rm FOPT}$	$(-12.66 \pm 5.07) * 10^{-3}\rm GeV^6$
$\mathcal{O}_6\|_{\rm CIPT}$	$(-15.96 \pm 3.03) * 10^{-3}\rm GeV^6$
$\mathcal{O}_6\|_{\rm DCIPT}$	$(-11.80 \pm 1.38) * 10^{-3}\rm GeV^6$
$\mathcal{O}_8\|_{\rm FOPT}$	$(15.20 \pm 5.06) * 10^{-3}\rm GeV^8$
$\mathcal{O}_8\|_{\rm CIPT}$	$(20.26 \pm 5.07) * 10^{-3}\rm GeV^8$
$\mathcal{O}_8\|_{\rm DCIPT}$	$(21.47 \pm 2.49) * 10^{-3}\rm GeV^8$

Conclusions

❋ We have developed a conceptually improved version of the CIPT approach to the hadronic τ-decays. The approximations to the Adler function within the new framework automatically satisfy the required cut-plane (Kallen-Lehmann) analyticity.

❋The quark-hadron duality is realized via the simple *ansatz* which relates the theory with the experimental data.

❋ The Advantage over the minimal analytic QCD model (APT) is that our approach does not invalidate the OPE for the current correlator. In fact the power suppressed contributions of ultraviolet origin are cancelled.

❋ Using the new framework we have presented a reasonable description of the hadronic τ-decays in the non-strange V-channel. We have **self-consistently** extracted from the 2005/2008 ALEPH V-channel data the numerical values for the strong coupling constant and the effective condensate combinations of dimensions $d = 4 - 12$. Reasonable numerical values for these parameters are predicted. In the Minkowski space, the lower boundary for the quark-hadron duality is estimated.

❋ The stability of the results with respect to perturbation theory corrections is examined.