Self-Adjoint extension procedure for scattering problems

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I . Introduction

Last years much attention is paid to the Self-Adjoint extension procedure in the Schrodnger and various relativistic equations for attractive inverse square

$1/r^2$ potentials at the origin **[1 P.Giri.Mod.Phys.Lett. A23,2177 (2008)],** because this interaction is realised in nature.For example Model of valence electron [**2 M.Eliashevich."Atomic and Molecular Spectroscopy". "Nauka".Moscow.1962 (Russian)]**, Coulomb and Hulten potential in the Klein-Gordon and Dirac equations [3 **H.Saad. math-ph/0709.4014(2007)],** Black holes [4 **K.Maharana. Int.J.Mod.Phys.A 22, 1717(2007)]**, Conformal Quantum Mechanics [**5. H.E.Camblong and C.R.Ordonez. Phys.Lett.A3,45 (2005)]** Aaronov-Bom effect[**6. J.Audretsh et all.J.Phys.A34:235 (2001).**], Dirac monopoles[**7.**

Yu.A.Sitenko et all.hep-th/0609053 (2006)], Quantum Holl effect [**8. P.Giacconi and R.Soldati.J.Phys.A33,5193(2000)],** Calogero model [**9.L.Feher et all.Nucl.Phys.B.715,713 (2005)**] and so on. In [**10. T.Nadareishvili, A.Khelashvili." Pragmatic SAE procedure in the Schrodinger equation for the inverse-square-like potentials''. arXiv:1209.2864]** we investigated SAE procedure for such kind of potentials for **bound** states. Now we perform SAE procedure for **scattering** problems.

II. Orthogonality condition for scattering problem and the SAE parameter

It was shown in our papers [**11. A.A.Khelashvili,T.P. Nadareishvili , Am.J.Phys.**79, **668 (2011);arXiv: 1009.2694v2; 12. A.A.Khelashvili, T.P. Nadareishvili,Bulletin of the Georgian National Academy of Sciences (Moambe). 6, 68 (2012); arXiv: 1102.1185v2]** that the following equation for the **full** radial function $R(r)$

$$-\frac{1}{2m}\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]R(r)+V(r)R(r)=ER(r)$$

(2.1)

is compatible with the 3-dimensional Schrodinger equation, if the **restriction**

$$\lim_{r\to 0} rR(r) = 0 \qquad (2.2)$$

is satisfied at the **origin** of coordinates. Moreover this restriction follows from the very general principles of quantum mechanics **both** for regular as well as for singular potentials **[11]**. Therefore below our consideration will be based on this equation, instead of **reduced** radial equation for $u(r) = rR(r)$. We begin with inverse square potential **[4-6]**

$$V = -\frac{V_0}{r^2};\ V_0 > 0 \qquad (2.3)$$

For general scattering solution of the Eq. (2.1) we have

$$R(r) = \sqrt{k/r}\{A(k)J_P(kr) + B(k)J_{-P}(kr)\};\ k^2 = 2mE; E > 0$$

(2.4)

Where $k^2 = 2mE;\ E > 0$ and

$$P = \sqrt{(l+1/2)^2 - 2mV_0}\ . \qquad (2.5)$$

Here $J_P(kr)$ is the Bessel function. For $0 < P < 1/2$ **both** terms in (2) obey (2.2) and we must **keep both** solutions! Moreover when we keep the **second** solution model's parameters obey the **restriction**

$$l(l+1) < 2mV_0 \qquad (2.6)$$

For scattering problem orthogonality condition is:

$$I = \int_0^\infty r^2 R_{k'}^*(r) R_k(r) dr = 2\pi\delta(k'-k) \qquad (2.7)$$

Now in (2..7) we use integrals for J_P and J_{-P} functions from [6] and get

$$I = \left\{AA^* + BB^* + (AB^* + A^*B)\cos\pi P\right\}\delta(k-k') +$$
$$+ \frac{2\sin\pi P}{\pi(k^2 - k'^2)}\left\{\left(\frac{k}{k'}\right)^P B^*(k')A(k) - \left(\frac{k}{k'}\right)^{-P} A^*(k')B(k)\right\} \quad (2.8)$$

From (2.7) and (2.8) we obtain orthogonality condition

$$\frac{B^*(k')}{A^*(k')}(k')^{-2P} = \frac{B(k)}{A(k)}k^{-2P} = \tau_S \qquad (2.9)$$

where τ_S is a SAE parameter in scattering case. Also from (2.7) and (2.8) we also obtain

$$AA^*\left[\tau_S^2 k^{4P} + 2\tau_S k^{2P}\cos\pi P + 1\right] = 2\pi \quad (2.10)$$

Based on the methodology of the article [**13.J.Audretsch at all.J.Phys.A28,2359(1995)**], where following integral

$$\lim_{r\to\infty}\int_0^r R_{k'}^*(r)R_k(r)r^2 dr = \frac{1}{k'^2 - k^2}\left[u_{k'}^*\frac{du_k}{dr} - u_k\frac{du_{k'}}{dr}\right]_0^r \quad (2.11)$$

is investigated one can show, that τ_S parameter is introduced from the **lower** limit of the (2.11) as it was for **bound** states **[10]** and (2.10) condition is introduced from the **upper** limit of this integral and has **no analogy** in bound states case.

III. Calculation of scattering partial amplitudes and phase shifts

For calculation of partial scattering l amplitudes let us write asymptotic expression of radial function R (2.4) at the infinity

$$\underset{r\to\infty}{R} \approx \frac{1}{\sqrt{2\pi}}\frac{1}{r}e^{ikr}\left[Ae^{-i\left(P+\frac{1}{2}\right)\frac{\pi}{2}}+Be^{i\left(P-\frac{1}{2}\right)\frac{\pi}{2}}\right]+$$

$$+\frac{1}{\sqrt{2\pi}}\frac{1}{r}e^{-ikr}\left[Ae^{i\left(P+\frac{1}{2}\right)\frac{\pi}{2}}+Be^{-i\left(P-\frac{1}{2}\right)\frac{\pi}{2}}\right] \qquad (3.1)$$

On the hand one has [**L.Landau;E.Lifchiz."Quantum Mechanics". "Nauka". Moscow.2002 (Russian)**]

$$\underset{r\to\infty}{R} \approx \frac{2}{r}\sin\left(kr-\frac{l\pi}{2}+\delta_l\right) \qquad (3.2)$$

and by comparison of (3.1) and (3.2) with using of (2.9), we obtain following partial scattering amplitude

$$S_l = e^{2i\left[l+\frac{1}{2}-P\right]\frac{\pi}{2}}\frac{1+\tau_S k^{2P}e^{i\pi P}}{1+\tau_S k^{2P}e^{-i\pi P}} \qquad (3.3)$$

Notice that in the (3.3) factor $\frac{1+\tau_S k^{2P} e^{i\pi P}}{1+\tau_S k^{2P} e^{-i\pi P}}$ is a **new** term. It arises, because we **kept second term** in the (2.4) or, equivalently we performed a SAE procedure. Eq.(3.3) gives physically **correct** results

a) When $\tau_S = 0$ or B=0, from (3.3) we obtain a standard result [**15. A.M. Perelomov, V.S. Popov. TMF.Vol 4.p48 (1970) (Russian)]**

$$S_l^{st} = e^{2i\left[l+\frac{1}{2}-P\right]\frac{\pi}{2}} \tag{3.4}$$

b) For $\tau_S = \pm\infty$ -or A=0, we get **additional** partial amplitude

$$S_l^{add} = e^{2i\left[l+\frac{1}{2}+P\right]\frac{\pi}{2}} \tag{3.5}$$

which is obtained from (3.4) by replacement $P \to -P$ and which is the common rule for transition from standard to additional states [10].

c)In [10] we have shown, that (2.3) potential has a **single** level (which appears owing the SAE procedure)

$$E = -\frac{2}{m}\left[-\frac{1}{\tau_B}\right]^{\frac{1}{P}}; 0 < P < 1/2 \tag{3.6}$$

where τ_B is a SAE parameter for bound states.

Now we obtain this level as a **pole** of (3.3) amplitude. It is obvious, that **standard** (3.4) amplitude have **no**

poles and so we have **no bound** states for (2.3) potential in the **usual** quantum mechanics [14, **16. A.Messia. "Quantum mechanics".Vol 1; "Nauka".Moscow.1970 (Russian)]**, when $\tau_S = 0$. So it is clear, that (3.3) may have **poles** due to the new factor

$$Q = \frac{1+\tau_S k^{2P} e^{i\pi P}}{1+\tau_S k^{2P} e^{-i\pi P}} \tag{3.7}$$

and indeed if we take $k = i\rho$ or $k^2 = -\rho^2 = 2mE; \quad (E<0)$ then (3.3) has **poles** at

$$\left[-\rho^2\right]^P \tau_S e^{-i\pi P} = -1 \tag{3.8}$$

and from which we **obtain** (3.6)! Now (3.3) can be written as

$$S_{INV,l} = e^{2i\delta_{INV,l}} \tag{3.9}$$

where INV indicates, that we consider scattering by (2.3) potential and

$$\delta_{INV,l} = \delta_{st} + \delta_{SAE} = \left[l + \frac{1}{2} - P\right]\frac{\pi}{2} + arctg\frac{\tau_S k^{2P}\sin\pi P}{1+\tau_S k^{2P}\cos\pi P} \tag{3.10}$$

where the **second term is a new one**

$$\delta_{SAE} = arctgX; \qquad X = \frac{\tau_S k^{2P}\sin\pi P}{1+\tau_S k^{2P}\cos\pi P} \tag{3.11}$$

and it follows thanks to the SAE procedure. Let us make some **remarks:**

1) $\delta_{INV,l}$ is **depended** on the energy($k^2 = 2mE$) and so **scale invariance is violated**. It will be **restored** only for **three** values of τ_S parameter: $\tau_S = 0$ and $\tau_S = \pm\infty$. Particularly for $\tau_S = 0$ (B=0) , from (3.10) we obtain a standard phase [15]

$$\delta_l^{st} = \left[l + \frac{1}{2} - P \right] \frac{\pi}{2} \tag{3.12}$$

and for $\tau_S = \pm\infty$ (A=0), we obtain **additional** states' phase

$$\delta_l^{add} = \left[l + \frac{1}{2} + P \right] \frac{\pi}{2} \tag{3.13}$$

2) We consider **attractive** (2.3) potential, for which $\delta_{INV,l} > 0$, but as it is clear from (3.11) due to the $arctg \dfrac{\tau_S k^{2P} \sin \pi P}{1 + \tau_S k^{2P} \cos \pi P}$ term, $\delta_{INV,l}$ may become **negative** and we get **repulsive** potential! We have **two** possibilities:

I. From the physical motivation **restrict** τ_S parameter so that this changing attractiion to repulsion **does not take place** or as see from (3.11) demand

$$\delta_{INV,l} = \left[l + \frac{1}{2} - P \right] \frac{\pi}{2} + arctg \frac{\tau_S k^{2P} \sin \pi P}{1 + \tau_S k^{2P} \cos \pi P} > 0 \tag{3.14}$$

and **restrict** parameters correspondingly.

II. We must assume that τ_S **change the NATURE of potential** and (3.14) isn't fulfilled!

Now we investigate changes, following by SAE procedure in the usual scattering theory relations [14]:

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l - 1)P_l(\cos\theta)\,;\; S_l = e^{2i\delta_l}\,;$$

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \quad \sigma_l = 4\pi(2l+1)\left|f_l\right|^2$$

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)f_l P_l(\cos\theta)\,;$$

$$f_l = \frac{1}{2ik}\left(S_l - 1\right) = \frac{e^{i\delta_l}\sin\delta_l}{k} \qquad (3.15)$$

As shown above **additional** solution **appear** for such l which satisfy the condition (2.6) or (3.15) relations **need SAE procedure** ($l = 0$ **always** need SAE procedure!). From (2.6) we get that this procedure **is necessary** for such l -s:

$$l = 0,1,2\ldots\left[-\frac{1}{2} + \sqrt{\frac{1}{4} + 2mV_0}\right] \qquad (3.16)$$

where[ ] denotes integer part of the $-\frac{1}{2} + \sqrt{\frac{1}{4} + 2mV_0}$ expression. So from (3.15),(3.16),we obtain

$$f(\theta)=\frac{1}{2ik}\left\{\sum_{l=0}^{\left[-\frac{1}{2}+\sqrt{\frac{1}{4}+2mV_0}\right]}(2l+1)P_L(\cos\theta)\left[e^{2i\left(\delta_{st}+arctgX\right)}-1\right]\right\}$$

$$+\frac{1}{2ik}\sum_{\left[-\frac{1}{2}+\sqrt{\frac{1}{4}+2mV_0}\right]}^{\infty}(2l+1)P_l(\cos\theta)\left[e^{2i\delta_{st}}-1\right]$$

(3.17)

Hence the scattering amplitude (3.17) can be written as

$$f(\theta)=f_{SAE}(\theta)+f_{INV}(\theta) \qquad (3.18)$$

where f_{SAE} denotes first term in (3.17) in which SAE procedure is **performed** and f_{INV} denotes second term without this procedure. As it is clear from (2.6) inequality in the limit $V_0 \to \infty$, **leading** term in (3.18) is f_{SAE}, and for small V_0, f_{INV}. For differential cross section we obtain

$$\frac{d\sigma}{d\Omega}=|f(\theta)|^2=|f_{INV}(\theta)|^2+|f_{SAE}(\theta)|^2+2\,\mathrm{Re}\,f_{INV}^*(\theta)f_{SAE}(\theta)$$

(3.19)

In (3.19) last **two terms** arises as a result of performing SAE procedure and **formally is similar to the short range potential.** So we again obtained that SAE procedure plays **important role** in interactions and it corresponds last years terminology - **"Self-adjoint extension Physics''[1,9]!**For small V_0 in f_{SAE} we **keep only** $l = 0$ term in (3.17) and we obtain

$$f(\theta) = \frac{1}{2ik}\left\{\left[e^{2i\left(\delta_{0,st} + arctgX\right)} - 1\right] + \sum_{l=1}^{\infty}(2l+1)P_l(\cos\theta)\left[e^{2i\delta_{l,st}} - 1\right]\right\}$$

(3.20)

and by using (3.11) and (3.15) finally we get

$$f(\theta) = \frac{\pi m V_0}{2k\sin(\theta/2)} + \frac{1}{k}e^{i\left[\frac{1}{2} - P\right]\pi} e^{iarctgX}\frac{X}{\sqrt{1+X^2}} \quad (3.21)$$

Notice, that in (3.21) **second SAE term** is **independent** on the θ angle, but if we consider $l \neq 0$ terms in the f_{SAE} it become **depending** on the θ angle. For differential cross section we obtain

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left\{\begin{aligned}&\frac{(\pi m V_0)^2}{4\sin^2(\theta/2)} + \frac{X^2}{1+X^2}\left[1 - \frac{\pi m V_0}{\sin(\theta/2)}\sin\left(\frac{1}{2} - P\right)\frac{\pi}{2}\right] \\ &+ \frac{X}{1+X^2}\cos\left(\frac{1}{2} - P\right)\frac{\pi}{2}\end{aligned}\right\}$$

(3.22)

(3.22) expression **differs** from standard differential cross section $\frac{d\sigma_{st}}{d\Omega}$ ($\tau_S = 0$) and this effect may be observed in the **slow particles scattering experiments, where most important are** $l=0$ **states.**

For small X ,from (3.11) definition and (3.22) we obtain

$$\frac{d\sigma}{d\Omega}=\frac{1}{k^2}\left\{\frac{(\pi m V_0)^2}{4\sin^2(\theta/2)}+\tau_S k^{2P}\sin\pi P\cos\left(\frac{1}{2}-P\right)\frac{\pi}{2}\right\}$$

(3.23)

we see **explicit** dependence of (3.23) on the τ_S parameter and the **demand,** that cross section is **positive, restricts** this parameter $\tau_S > -\frac{(\pi m V_0)^2}{4\sin^2(\theta/2)}\sin\pi P\cos\left(\frac{1}{2}-P\right)\frac{\pi}{2}$ .It is known, that total cross section σ for arbitrary V_0 for the (2.3) potential in usual quantum mechanics ($\tau_S=0$) is infinite [14], because this potential decreases at infinity more slowly then $1/r^{2+\varepsilon}$,but in (3.22) for negative X or τ_S last term is negative and total cross section σ can become finite! We must recognize that τ_S change the NATURE of the potential or we should neglect such values of τ_S parameter!

IV. Scattering length and effective radius

In the [15] article scattering length in the $l = 0$ state for the potential

$$V(r) = -\frac{V_0}{r^2}\theta(R-r) \qquad (4.1)$$

is calculated

We now obtain **more general formula using SAE.** When $r < R$ wave function has following form

$$\chi_0 = rR_0(r) = \begin{cases} Ar^{\frac{1}{2}+P} + Br^{\frac{1}{2}-P}; 2mV_0 < \frac{1}{4}; P = \sqrt{\frac{1}{4} - 2mV_0} \\ r^{\frac{1}{2}} \sin(\nu \ln r + \gamma_0); 2mV_0 > \frac{1}{4}; \nu = \sqrt{2mV_0 - \frac{1}{4}} \end{cases}$$

(4.2)

where γ_0 is well known SAE parameter in the case of particle's **''falling''** on the centre [15], and $Br^{\frac{1}{2}-P}$ is a new term. For $r > R$

$$\chi_0 = C(r-a) \qquad (4.3)$$

where a is scattering length. Condition of "**Sewing**" at $r = R$ gives

$$a = -R\frac{(1-2P)AR^{P} + BR^{-P}(1+2P)}{(1+2P)AR^{P} + BR^{-P}(1-2P)};\ 2mV_0 < \frac{1}{4}$$

(4.4)

$$a = -R\frac{1-2\nu ctg(\nu \ln R - \gamma_0)}{1+2\nu ctg(\gamma \ln R - \gamma_0)};\ 2mV_0 > \frac{1}{4}$$

(4.5)

When $B = 0$,we get result of [15]

$$a = -R\frac{1-2P}{1+2P} \qquad (4.6)$$

It is clear from (4.6) that as $P < 1/2$, $a < 0$ and it corresponds to **attractive** (4.1) potential [15], but from (4.4) we have **no definite sign** of the a .

Now SAE parameter is $\tau = B/A$ and

$$a = \frac{a_1\tau + b_1}{a_2\tau + b_2};\ a_1 = -(1+2P)R^{1-P};$$

$$b_1 = -(1-2P)R^{1+P};\ a_2 = (1-2P)R^{-P};$$

$$b_2 = (1+2P)R^{P} \qquad (4.7)$$

In the region

$$\tau_1 < \tau < \tau_2\,; \qquad (4.8)$$

$$\tau_1 = \frac{1+2P}{2P-1}R^{2P} < 0;\ \tau_2 = \frac{2P-1}{1+2P}R^{2P} < 0;\ 0 < P < 1/2$$

$a > 0$ - potential becomes **repulsive!** Again we have **two** alternatives: a) From the physical motivation **exclude** (4.8) region or b) Agree that τ can **change interaction nature!**

Remarks:

1) If we expand (4.4) and (4.5) near $2mV_0 = 1/4$,obtain **relation between** γ_0 and τ

$$2ctg\gamma_0 = \frac{1-\tau}{1+\tau};$$

2)Total cross section $\sigma_{tot} = 4\pi a^2(\tau) > 0$ demand **restrict** τ parameter .

Now perform SAE procedure for the (4.1) potential in the $2mV_0 < \frac{1}{4}$ region for **scattering effective radius** formulae [14,16]

$$r_0 = 2\int_0^\infty \left[u_0^2(r) - \chi_0^2(r)\right]dr$$

$$u_0 = C(r-a) \qquad (4.9)$$

where

$$\chi_0 = \begin{cases} Ar^{\frac{1}{2}-P} + Br^{\frac{1}{2}-P}; r < R \\ C(r-a); r > R \end{cases} \qquad (4.10)$$

Now $Br^{\frac{1}{2}-P}$ is a **new** term in the (4.10) and we obtain

$$r_0 = \frac{2}{3}C^2\left\{(R-a)^3 + a^3\right\} - D^2\left\{\frac{\tau^{-2}}{2(P+1)}R^{2P+2} \frac{R^{2(1-P)}}{2(1-P)} - \tau^{-1}R^2\right\}$$

(4.11)

where SAE parameter τ is defined above, c is normalization constant and $D = \dfrac{C(R-a)}{R^{\frac{1}{2}-P} + \tau^{-1}R^{\frac{1}{2}+P}}$

Again $r_0 > 0$ demand **restrict** τ parameter and now total cross section depends on τ as

$$\sigma_{tot} = \frac{4\pi a^2(\tau)}{1 + a(\tau)\left[a(\tau) - r_0(\tau)\right]k^2 + \left[\frac{1}{2}ar_0(\tau)\right]^2 k^4}; \qquad k^2 = 2mE$$

(4.12)

V.Modification of Rutheford formula

Now we consider scattering problem for model of valence electron

$$V = -\frac{V_0}{r^2} - \frac{\alpha}{r};\left(V_0, \alpha > 0\right) \qquad (5.1)$$

This potential ‘’naturally’’ **appears** in the Klein-Gordon equation for Coulomb potential. Now following the [**17.W.Krolikowski;Bulletin De Lacademics polonaise.Vol 18,83 (1979). 18.A.A.Khelashvili, T.P.Nadareishvili.Bulletin of Georgian Acad.Sci.Vol 164 N1 (2001)**] articles, we get

$$R(r) = C_1 \rho^{-1/2+P} e^{-\frac{\rho}{2}} {}_1F_1(1/2+P-\lambda, 1+2P, \rho) +$$

$$+ C_2 \rho^{-1/2-P} e^{-\frac{\rho}{2}} {}_1F_1(1/2-P-\lambda, 1-2P, \rho)$$

(5.2)

where P is given by (2.5) and

$$\rho = 2ikr; \; \lambda = -i\frac{m\alpha}{k} = -i\eta; \; k = \sqrt{2mE}; \; E > 0;$$

$$\eta = \frac{m\alpha}{k} \qquad (5.3)$$

From the behaviour of (5.2) wave function at the origin and (2.9) definition, now τ_S parameter

$$\tau_S = \frac{C_2(k)}{C_1(k)}(2ik)^{-2P} \tag{5.4}$$

and for partial scattering partial amplitude we get

$$S_{VE} = e^{2i\left[l+\frac{1}{2}-P\right]\frac{\pi}{2}+2i\delta^{st}_{Coulomb}} \frac{1+\tau_S(2ik)^{2P}We^{i\left(\delta^{add}_{Coulomb}-\delta^{st}_{Coulomb}\right)}}{1+\tau_S(2ik)^{2P}We^{-2i\pi P}e^{i\left(\delta^{add}_{Coulomb}-\delta^{st}_{Coulomb}\right)}}$$

(5.5)

where

$$W = \frac{\Gamma(1-2P)}{\Gamma(1+2P)} \frac{\left|\Gamma\left(\frac{1}{2}+P+\lambda\right)\right|}{\left|\Gamma\left(\frac{1}{2}-P+\lambda\right)\right|};$$

$$\delta^{st}_{Coulomb} = \arg\Gamma\left(\frac{1}{2}+P+\lambda\right); \delta^{add}_{Coulomb} = \arg\Gamma\left(\frac{1}{2}-P+\lambda\right)$$

(5.6)

In the (5.5) expression $\frac{1+\tau_S(2ik)^{2P}We^{i\left(\delta^{add}_{Coulomb}-\delta^{st}_{Coulomb}\right)}}{1+\tau_S(2ik)^{2P}We^{-2i\pi P}e^{i\left(\delta^{add}_{Coulomb}-\delta^{st}_{Coulomb}\right)}}$ is a **new term** and it arises, because we **kept** the **second** term in the (5.2). Eq.(5.5) gives physically **correct** results in cases:

a) When $\tau_S = 0$ or $C_2 = 0$, from (5.2) we obtain standard result [15] $S^{st}_{VE} = e^{2i\left[l+\frac{1}{2}-P\right]\frac{\pi}{2}+2i\delta^{st}_{Coulomb}}$

b) For $\tau_S = \pm\infty$ -or $C_1 = 0$ we get **additional** partial amplitude $S_{VE}^{st} = e^{2i\left[l+\frac{1}{2}+P\right]\frac{\pi}{2}+2i\delta_{Coulomb}^{add}}$

c) In [10] we obtain following eigenvalue transcendental equation for (5.1) potential

$$\frac{\Gamma(1/2-\lambda-P)}{\Gamma(1/2-\lambda+P)} = -\tau(-8mE)^P \frac{\Gamma(1-2P)}{\Gamma(1+2P)}$$

(5.7)

It is easy to show, that (5.7) equation gives **a pole** of (5.5) amplitude. Now (5.5) can be rewritten as

$$S_{VE} = e^{2i\delta_{VE}} \quad ;$$

$$\delta_{VE} = \delta_l^{st} + \delta_{Coulomb}^{st} + \delta_{SAE} \qquad (5.8)$$

VE indicates, that scattering is by (5.1) valence electron potential and the δ_{SAE} is a **new term**

$$\delta_{SAE} = arctgZ$$

$$Z = \frac{\tau_S (2k)^{2P} W \sin\left(\pi P + \delta_{Coulomb}^{add} - \delta_{Coulomb}^{st}\right)}{1 + \tau_S (2k)^{2P} W \cos\left(\pi P + \delta_{Coulomb}^{add} - \delta_{Coulomb}^{st}\right)} \qquad (5.9)$$

and it arises, because we **performed SAE procedure.** Now we investigate **changes** by SAE procedure in the **usual** Rutheford formula. We again obtain (3.17), but in the exponents expression (5.9) appears. For small V_0 in f_{SAE} we keep only $l=0$ term in (3.17) and obtain

$$f(\theta)=\frac{1}{2ik}\left\{\begin{array}{l} e^{2i\left\{\left[\frac{1}{2}-P\right]\frac{\pi}{2}+\delta^{st}_{0,Coulomb}\right\}}\left[e^{2iarctgZ}-1\right]+ \\ +\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\left[e^{2i\left\{\left[l+\frac{1}{2}-P\right]\frac{\pi}{2}+\delta^{st}_{l,Coulomb}\right\}}-1\right]\end{array}\right\}$$

(5.10)

Second term is **standard** scattering amplitude and for small V_0 is calculated in the monograph [**19. H.M. Pilkun. ''Relativistic Particle Physics''. ''Mir''.Moscow. 1983**] for **Coulomb scattering** in the Klein –Gordon equation, but as we mentioned above (5.1) potential appears in the Klein-Gordon equation for Coulomb potential. Therefore we obtain for second term

$$f^0_{VE}(\theta)=f_{Coulomb}(\theta)\left\{1-\pi V_0 k\sin\frac{\theta}{2}e^{2i[\sigma_{-1/2}-\sigma_0]}\right\}$$

(5.11)

where $f_{Coulomb}(\theta)$ is Rutheford scattering amplitude [19]

$$f_{Coulomb}(\theta) = -\frac{\eta}{2k\sin^2\frac{\theta}{2}} e^{\left\{-i\eta\ln\left(\sin^2\frac{\theta}{2}\right)+2i\sigma_0\right\}}$$

;

$$e^{2i\sigma}{}_0 = \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}; \qquad e^{2i\sigma}{}_{-1/2} = \frac{\Gamma(1/2+i\eta)}{\Gamma(1/2-i\eta)}$$

(5.12)

Finally we obtain

$$f(\theta) = -\frac{\eta}{2k\sin^2\frac{\theta}{2}} e^{\left\{-i\eta\ln\left(\sin^2\frac{\theta}{2}\right)+2i\sigma_0\right\}} \left\{1 - \pi V_0 k \sin\frac{\theta}{2} e^{2i[\sigma_{-1/2}-\sigma_0]}\right\} +$$

$$+\frac{1}{k} e^{2i\left\{\left[\frac{1}{2}-P\right]\frac{\pi}{2}+\delta^{st}_{0,Coulomb}\right\}} e^{iarctgZ} \frac{Z}{\sqrt{1+Z^2}}$$

(5.13)

In (5.13) the second SAE term is **independent** on the θangle (for **big** angles θ, this fact becomes **important**, because for such angles first Rutheford term in (5.13) is **small!**) but if we consider $l \neq 0$ terms in the f_{SAE} it will be depended on the θ angle. For differential cross section we get

$$\frac{d\sigma}{d\Omega}=\frac{\eta^2}{4k^2\sin^4\frac{\theta}{2}}\left\{1+(\pi V_0 k)^2\sin^2\frac{\theta}{2}-2\pi V_0 k\sin\frac{\theta}{2}\cos 2\left(\sigma_{-\frac{1}{2}}-\sigma_0\right)\right\}+$$

$$+\frac{1}{k^2}\frac{Z^2}{1+Z^2}\left[1+\frac{\eta}{k^2\sin^2\frac{\theta}{2}}\left(\sin\alpha_1-\pi V_0 k\sin\frac{\theta}{2}\sin\alpha_2\right)\right]-\frac{\eta}{k^2\sin^2\frac{\theta}{2}}\frac{Z}{1+Z^2}\left[\cos\alpha_1-\pi V_0 k\sin\frac{\theta}{2}\cos\alpha_2\right]$$

(5.14)

where

$$\alpha_1=(1/2-P)\pi+2\delta^{st}_{0,Coulomb}+\eta\ln\sin^2\frac{\theta}{2}-2\sigma_0;$$

$$\alpha_1=(1/2-P)\pi+2\delta^{st}_{0,Coulomb}+\eta\ln\sin^2\frac{\theta}{2}-2\sigma_{-\frac{1}{2}}$$

(5.15)

(5.14) **differs** from standard differential cross section $\frac{d\sigma_{st}}{d\Omega}$ (when $\tau_S=0$) and this effect **may be observed in the slow particles scattering experiments, where most important are $l=0$ states.** Now as in (3.22), in (5.14) for positive Z or τ_S last term is **negative** and total cross section σ can become **finite!**

For small Z, from (5.9) definition and (5.15) we obtain

$$\frac{d\sigma}{d\Omega}=\frac{\eta^2}{4k^2\sin^4\frac{\theta}{2}}\left\{1+(\pi V_0 k)^2\sin^2\frac{\theta}{2}-2\pi V_0 k\sin\frac{\theta}{2}\cos 2\left(\sigma_{-\frac{1}{2}}-\sigma_0\right)\right\}-$$

$$-\frac{\eta}{k^2\sin^2\frac{\theta}{2}}\tau_S(2k)^{2P}W\sin\left(\pi P+\delta^{add}_{Coulomb}-\delta^{st}_{Coulomb}\right)\left[\cos\alpha_1-\pi V_0 k\sin\frac{\theta}{2}\cos\alpha_2\right]$$

(5.16)

We see an explicit dependence of cross section on the τ_S parameter and now second SAE term depends on θ angle in difference of (3.23).

VIII. Summary. Outlook

1.We show, that for $\lim_{r \to 0} r^2 V \to -V_0$ potentials it is **necessary to keep second additional** solutions in the 0<P<1/2 interval and to introduce SAE τ parameter for scattering problems.

2. We also show, that physical quantities E, a, r_0, σ **depend** on the τ parameter and the physical picture is **different** as compare of **usual** quantum mechanics. (SAE **can change nature** of the potential, δ phase becomes **energy depended** for $V = -V_0 / r^2$ potential and so on.

3. For the potential $V(r) = -\theta(R-r)V_0 / r^2$ scattering lengths are **modified** as well as the effective radius formulae

4. Rutherford Formula is also **modified** and the SAE procedure formally is similar to the **introduction of the short range potential** .

We have **three** possibilities:
I. It should be found **another strong requirement,** which ‘’destroys’’ additional states!

II. If it isn’t possible,try to**’’struggle’’** against τ parameter by physical demands: $r_0 > 0$, $\sigma > 0$, don’t change physical nature of potential and so on.

III. Admit SAE procedure and find **new** levels, cross sections and so on in the **future experiments!**

Now isn’t clear why the NATURE ’’select ’’ only standard quantum mechanics $(\tau = 0)$

THANK YOU FOR PATIENCE AND ATTENTION!!!